Conceptual Understanding + Reasoning = Fluency: The role of Big Ideas

Keynote presentation to the Annual Conference of the Mathematical Association of Tasmania, May 2015

by Professor Dianne Siemon
RMIT University

Overview:

- Understanding the problem
- Mathematical proficiency the goal
- The big ideas in number
- Multiplicative Thinking – a very BIG IDEA
- What works and where to we go to from here – implications for planning

It took 3 men 3 days to paint the inside of the house. How long will it take 2 men?

Understanding the problem

44% or 5.1 million jobs at risk from digital disruption [1]

The average performance of Year 8 students in mathematics has not changed since TIMSS 1995 [2]

Shifting 1% of workforce into STEM roles would add $57.4 billion to GDP over 20 years [1]

More than 20% of Year 8 students were being taught by mathematics by teachers who reported feeling only "somewhat" confident in teaching the subject [2]

37% of Australian Year 8 students did not achieve the Intermediate international benchmark (the minimum proficient standard expected) [2]

75% of the fastest growing occupations require STEM [1]

Number of Year 12 students studying STEM subjects is declining [1]

1. Price-Waterhouse Report (April, 2015). A Smart Move: Future proofing Australia's workforce by growing skills in science, mathematics, engineering and maths (STEM)
2. Thompson, S., Hillman, K. & Wernet, N. (2012). Monitoring Australian Year 8 student achievement internationally: TIMSS 2011. Melbourne: ACER

Understanding the problem

The number of students taking intermediate and advanced maths at secondary school has fallen by 34% over the last 18 years [3]

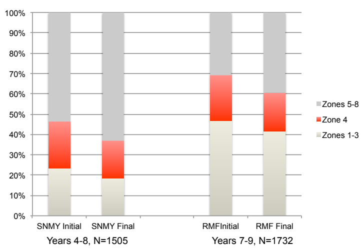
Interpreting, applying and evaluating mathematical outcomes is an area of relative strength for Australian 15-year olds but **formulating situations mathematically** and **employing mathematical concepts, facts, procedures and reasoning are areas of weakness [4]**

Australia's mean mathematical literacy performance declined significantly between PISA 2003 and PISA 2012 and males significantly outperformed females [4]

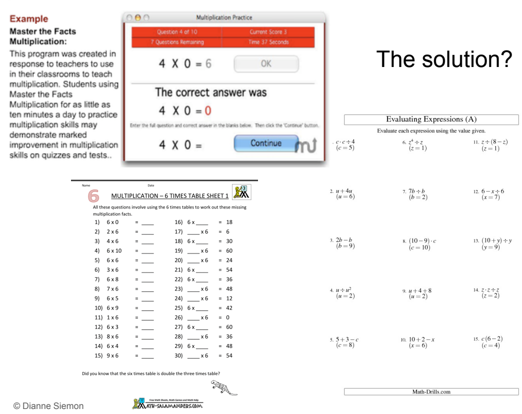
20% of mathematics and physics teachers are teaching out-of-field [5]

3. The Australian Industry Group (March, 2015). Progressing STEM skills in Australia. Melbourne: AiGroup
4. Thompson, S.De Bortoli, L. & Buckley, S. (2013). PISA 2012: How Australia measures up. Melbourne: ACER
5. Weldon, R. (March, 2015). Policy Insights. The teacher workforce in Australia: Supply and Demand Issues

Understanding the problem

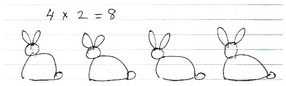


Access to multiplicative thinking a key aspect
Targeted teaching makes a difference



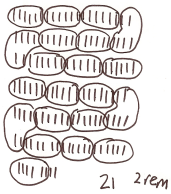
The solution?

Does it work?



A Year 8 student's response to the question: What sort of maths do you really like doing?

A Year 7 student's response to the problem: 128 ÷ 6 = ☐



Note the reliance on a count of groups, that is, the **equal groups** idea for multiplication and the corresponding **quotition** idea for division

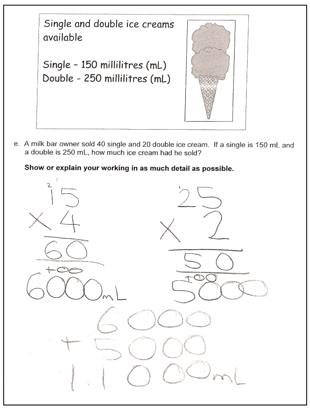
This is perpetuated in interpretations such as 3x = x + x + x and $2 \div \frac{1}{2}$ as 'how many halves in 2"

Big Ideas:
- Trusting the Count
- Place Value
- Multiplicative Thinking

Fluency without understanding

At what cost?

Big Ideas:
- Place Value
- Multiplicative Thinking


Single and double ice creams available

Single – 150 millilitres (mL)
Double - 250 millilitres (mL)

e. A milk bar owner sold 40 single and 20 double ice cream. If a single is 150 mL and a double is 250 mL, how much ice cream had he sold?

Show or explain your working in as much detail as possible.

15 x 4 = 60 +00 6000mL

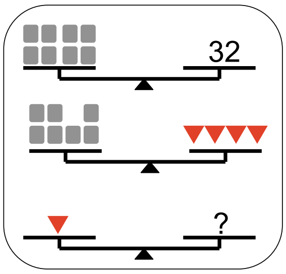
25 x 2 = 50 +00 5000

6000 + 5000 = 1.1000mL

What is needed? What is exercised?

1 x 4 = 4
2 x 4 = 8
3 x 4 = 12
4 x 4 = 16
5 x 4 = 20
6 x 4 = 24
7 x 4 = 28
8 x 4 = 32
9 x 4 = 36
10 x 4 = 40
11 x 4 = 44
12 x 4 = 48

9☐
x 4
392



www.criticalthinking.com/balance-math-and-more.html

"4 of anything is double double"

… facts at any cost versus mental strategies, reasoning…

What is needed? What is exercised?

Big Ideas:
- Trusting the count (equivalence)

…. formulating situations mathematically, employing mathematical facts, procedures and reasoning (PISA)…

A Mathematical Tug-of-War

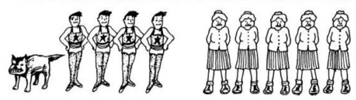
The problem is to use the information given to figure out who will win the third round in a tug-of-war.

Round 1: On one side are four acrobats, each of equal strength. On the other side are five neighborhood grandmas, each of equal strength. The result is dead even.

Round 2: On one side is Ivan, a dog. Ivan is pitted against two of the grandmas and one acrobat. Again, it's a draw.

Round 3: Ivan and three of the grandmas are on one side and the four acrobats are on the other.

Who will win the third round? Write an explanation of your reasoning.



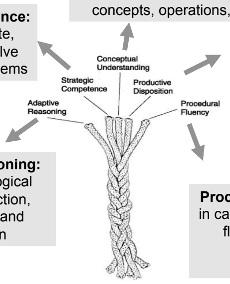
http://edublog.scholastic.com/post/launching-school-year-logic

Mathematical Proficiency – the ultimate goal

Conceptual understanding: Comprehension of mathematical concepts, operations, and relations

Strategic competence: Ability to formulate, represent, and solve mathematical problems

Productive disposition: Habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy



Adaptive reasoning: Capacity for logical thought, reflection, explanation, and justification

Procedural fluency: Skill in carrying our procedures flexibly, accurately, efficiently, and appropriately

Kilpatrick, J., Swafford, J. & Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics.* Washington DC: National Academy Press

The Proficiency Strands of the ACM

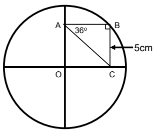
- ***Understanding*** (conceptual understanding) students build robust knowledge of adaptable and transferable mathematical concepts, make connections between related concepts and develop the confidence to use the familiar to develop new ideas, and the 'why' as well as the 'how' of mathematics.
- ***Fluency*** (procedural fluency) students develop skills in choosing appropriate procedures, carrying out procedures flexibly, accurately, efficiently and appropriately, and recalling factual knowledge and concepts readily.
- ***Problem solving*** (strategic competence) students develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively.
- ***Reasoning*** (adaptive reasoning) students develop increasingly sophisticated capacity for logical thought and actions such as analysing, proving, evaluating, explaining, inferring, justifying, and generalising.

http://www.australiancurriculum.edu.au/mathematics/content-structure

Conceptual understanding

What is involved in solving the following

Ariana had a goal-shooting average of 12 goals before the finals? In the semi-final she scored 18 goals and in the final she scored 15 goals. What was her end-of-season average?



O is the centre of a circle of diameter 17 cm. The figure ABC is a right triangle with the dimensions shown.

What is the length of the line AC?

.... connections between related concepts, confidence to use the familiar to develop new ideas (ACARA) …

Conceptual understanding

What is involved in solving the following problems?

35 feral cats were estimated to live in a 146 hectare nature reserve.
27 feral cats were estimated to live in another nature reserve of 103 hectares. Which reserve had the biggest feral cat problem?

At the present time a father is 3 times as old as his son.
In ten years time, he will be exactly twice his son's age.
How old are they at the present time?

.... formulating situations mathematically, employing mathematical facts, procedures and reasoning (PISA) …

Procedural fluency

What is involved in solving the following problems?

Samantha's Snail
Samantha's snail covered 1.59 metres in 6 minutes. How far might Samantha's snail travel in 17 minutes (in metres)?

Bargain Price
Juli bought a dress in an end-of-season sale for $49.35. The original price was covered by a 30% off sticker, but the sign on top of the rack said "Now an additional 15% off already reduced prices". How could she work out how much she had saved? What percentage of the original cost did she end up paying?

Big Ideas:
- Multiplicative Thinking
- Partitioning
- Proportional Reasoning

Procedural fluency

Year 4: Use equivalent number sentences involving addition and subtraction to find unknown quantities*

34 + 57
61 – 37
42 – 9.2

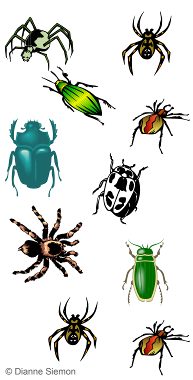
26 + ☐ = 20 + 17

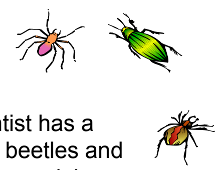
Year 5: Use equivalent number sentences involving multiplication and division to find unknown quantities*.

48 x 6
7 x 27

* From the Number and Algebra Strand of the Australian Mathematics Curriculum

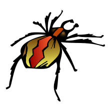
Problem solving





A mad scientist has a collection of beetles and spiders in a special container. The floor sensor tells her that there are 174 legs altogether and the infra-red detector tells her that there are 26 bodies. How many beetles and how many spiders are there?

Problem



Realise you have a problem – no immediately available solution strategy

PROBLEM TYPE

Process

Re read the problem, record the data what is required, UNDERSTAND

THINK about the problem, have I seen something like this before

TRY something

CHECK to see if it worked or not, does it satisfy the problem conditions

ASK
THINK → **DO**

Strategies

Guess and check or trial and error

Draw a diagram

Use equations

If … then reasoning

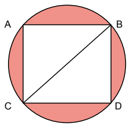
Table of values

Mathematical Reasoning

What is involved in the following?

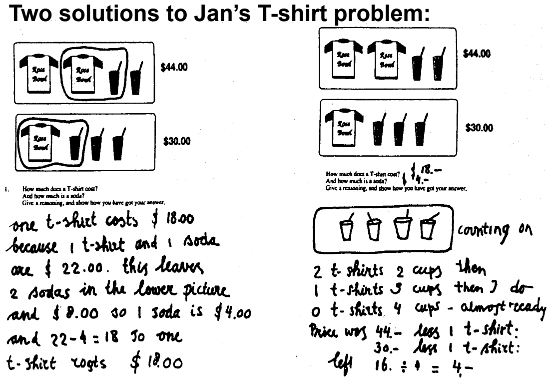
Sam and Joseph each had a shorter sister, and they argued about who was taller than his sister. Sam won the argument by 14 centimeters. He was 186 cm tall; his sister was 87 cm; and Joseph was 193 cm tall. How tall was Joseph's sister? [adapted from Thompson, 1993]

		6				5		
7	3		2	9				
5					3		2	
	2							9
		1	5		9	3		
6							7	
	6		4					5
				3	7		8	4
		7				1		



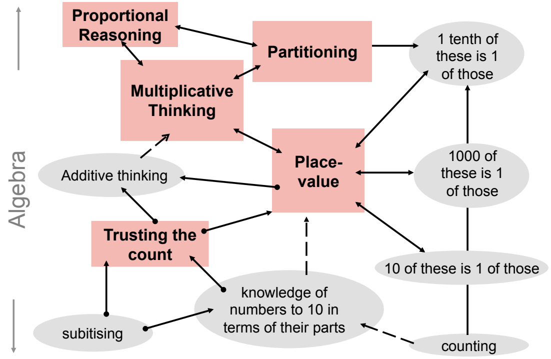
Find the area of the shaded part if the diameter of the circle is 22 cm and ABCD is a square

Two solutions to Jan's T-shirt problem:



(De Lange,1996)

The 'BIG IDEAS' in number



What is a 'big idea'?

- An idea, strategy, or way of thinking about some key aspect of mathematics, **without which students' progress in mathematics will be seriously impacted**
- Encompasses and **connects many other ideas** and strategies
- Provides an **organising structure** or a frame of reference that supports further learning and generalisations
- Cannot be clearly defined but can be **observed in activity** …

Trusting the Count

Is evident when children:

- know that counting is an **appropriate response** to questions which ask how many;
- believe that counting the same collection again will always produce the **same result** irrespective of how the objects in the collection are changed or manipulated;
- have **access to mental objects** for each of the numbers to ten (based on *part-part-whole knowledge*) that can be used flexibly without having to make, count or see these collections physically,
- demonstrate a **sense of numbers beyond ten**, and
- are able to **use small collections as units** when counting larger collections.

(Siemon, 2005)

Place Value

It is evident when students are able to:

- **recognise place value parts as units** (10 of these is 1 of those, 1 tenth of these is 1 of those)
- **model and/or represent** numbers in appropriate ways (manipulatives, number lines, number expanders, diagrams, exponents etc);
- **name** (hear, say, read, and write in words);
- **record** (write in numerals); and
- **consolidate** by **comparing, ordering, counting forwards and backwards in place-value parts**, and **renaming** numbers in in multiple ways

4003
\- 1628

399 13
~~4003~~
\- 1628

It's the numbers that move not the decimal point!!

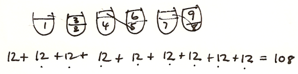
Multiplicative Thinking – a very BIG IDEA

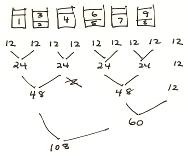
- a capacity to **work flexibly and efficiently with an extended range of numbers** including fractions, decimals and percents;
- an ability to **recognise and solve a range of problems involving multiplication or division** including direct and indirect proportion, rate and ratio; and
- the **means to communicate this effectively in a variety of ways** (e.g., words, diagrams, symbolic expressions, and written algorithms).

A muffin recipe requires 2/3 of a cup of milk. Each recipe makes 12 muffins. How many muffins can be made using 6 cups of milk?

Three solutions:

A muffin recipe requires 2/3 of a cup of milk. Each recipe makes 12 muffins. How many muffins can be made using 6 cups of milk?





Solutions which rely on counting all groups are essentially additive.

1 2/3 cup makes 12
1/3 cup makes 6
1 cup makes 18
So 6 cups makes 108
18 × 6 = 108

Solutions which rely on some form of proportional reasoning are essentially multiplicative.

From the Scaffolding Numeracy in the Middle Years (SNMY) Project

SNMY Extended Task

BUTTERFLY HOUSE...

Some children visited the Butterfly House at the Zoo.

They learnt that a butterfly is made up of 4 wings, one body and two feelers.
While they were there, they made models and answered some questions.
For each question, explain your working and your answer, in as much detail as possible.

a. How many wings, bodies and feelers would be needed for 7 model butterflies?
______ wings
______ bodies
______ feelers

b. How many complete model butterflies could you make with 16 wings, 4 bodies and 8 feelers?

Adapted from 'Butterflies and Caterpillars' (Kenney, Lindquist & Heffernan, 2002) for the SNMY Project (2003-2006)

This task had 9 items altogether including:

Items of **increasing complexity**, eg, "How many complete model butterflies could you make with 29 wings, 8 bodies and 13 feelers?"

Items involving **simple proportion and rate**, eg, "To feed 2 butterflies, the zoo needs 5 drops of nectar per day. How many drops would be needed per day to feed 12 butterflies?" …and

Items involving the **Cartesian product**, eg, given 3 different body colours, 2 types of feelers and 3 different wing colours, "How many different model butterflies could be made?"

SNMY Short Task

ADVENTURE CAMP ...

Camp Reefton offers 4 activities. Everyone has a go at each activity early in the week. On Thursday afternoon students can choose the activity that they would like to do again. The table shows how many students chose each activity at the Year 5 camp and how many chose each activity at the Year 7 camp a week later.

	Rock Wall	Canoeing	Archery	Ropes Course
Year 5	15	18	24	18
Year 7	19	21	38	22

Camp Reefton Thursday Activities

Reading and interpreting quantitative data relative to context

Open-ended question

a. What can you say about the choices of Year 5 and Year 7 students?

Recognising relevance of proportion

Problem solving, solution strategy unclear

b. The Camp Director said that canoeing was more popular with the Year 5 students than the Year 7 students. Do you agree with the Director's statement? **Use as much mathematics as you can to support your answer.**

Mathematics used, eg, percent, fractions, ratio

SNMY Project (2003-2006)

Targeted teaching works

For example, students in an identified sub-sample of 'at-risk' students within the SNMY Project demonstrated major shifts in achievement against the *Learning and Assessment Framework for Multiplicative Thinking* (LAF) as a result of an 18 week, 2 sessions per week teaching program* (Margarita Breed, PhD study)

Participants:	9 Year 6 students identified at Level 1 of the Framework in May 2004
Results:	All 9 students achieved at Level 4 or 5 of the Framework in November 2005

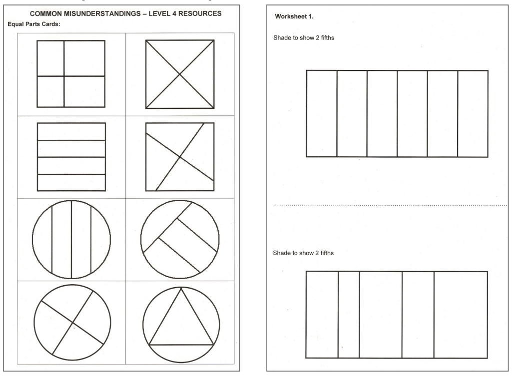
* A copy of the *Intervention Teaching Program for At Risk Students* is included in the *SNMY Project Findings, Materials and Resources* available on the DEECD and TasEd websites.

Partitioning

Is evident when students:

- use halving or related strategies to locate common fractions on a number line;
- estimate thirds and fifths by building on what is known (e.g., a third is smaller than a half … a fifth is smaller than a quarter);
- construct and use fraction models to rename, compare and order fractions; and
- reason multiplicatively to construct decimal representations on an open number line.

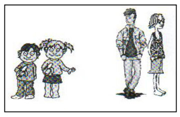
For example, The Equal Parts Tool



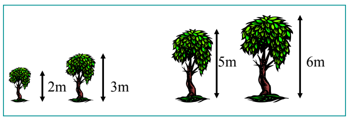
Proportional Reasoning

(a) … Who grew faster … Amy or Richard*?

(b) .. Which tree grew more from 2012 to 2015?

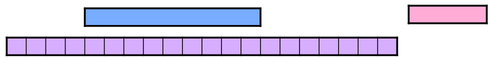


Adapted from Lamon (1988)



(c) … If the red rod is 1, what would the yellow rod be? … How many times longer is the yellow rod than the red rod? ..

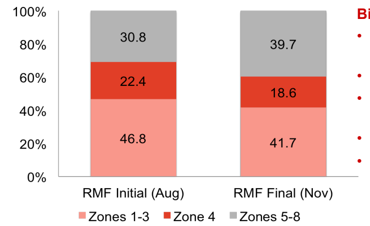
(d) The purple rod is 2. What fraction name would you give to the blue rod? …What fraction is the pink rod of the blue rod?



From the Assessment for Common Misunderstandings

BIG IDEAS – the key to STEM

Reframing Mathematical Futures – an AMSPP Priority Project 2013



Big Ideas:

- Proportional reasoning
- Partitioning
- Multiplicative thinking
- Place value
- Trusting the count

Percentage of students at Zones 1-3, 4 and 5-8 of the SNMY Learning and Assessment Framework, August and November 2013 (n=1532)

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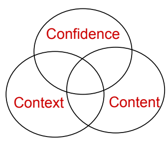
What works - Jo Boaler

- Case studies of 2 schools over 3 years with alternative mathematical teaching approaches: traditional, textbook approach versus open-ended activities at all time.
- Students who followed a traditional approach developed a **procedural knowledge that was of limited** use to them in unfamiliar situations.
- Students who learned mathematics in an open, project-based environment developed a **conceptual understanding that provided them with advantages** in a range of assessments and situations.
- The project students had been "apprenticed" into a system of thinking and using mathematics that helped them in both school and non-school settings.

Boaler, J. (1998), Open and Closed Mathematics: Student Experiences and Understandings, *Journal for Research in Mathematics Education, 29* (1), 41-62

Connecting the Big Ideas and Proficiencies

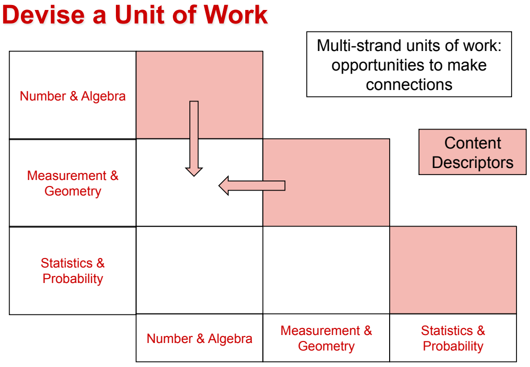


Work in planning teams to identify learning activities and/or assessment tasks in each cell

BIG IDEAS	Conceptual Understanding	Procedural Fluency	Problem Solving	Mathematical Reasoning
Trusting the Count				
Place Value				
Multiplicative Thinking				
Partitioning				
Proportional Reasoning				

Open ended questions
Rich Tasks
Investigations
Fermi Problems

Devise a Unit of Work



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Conclusion

The content descriptors of the ACM have the potential to support the development of the BIG IDEAS.

But the extent to which this potential is realised is heavily dependent on how the descriptors are **interpreted**, **represented, considered** and **connected** in practice.

Content descriptors do need to be in a form that is clearly assessable but, if these are taught and assessed in isolation with little attention to **student's prior knowledge** and the **underpinning ideas and strategies**, there is a substantial risk that access to the big ideas and multiplicative thinking in particular, will continue to elude many.

On the other hand, if the content descriptors are taught and assessed in conjunction with the proficiencies, that is,

- conceptual understanding,
- procedural fluency,
- mathematical reasoning and
- mathematical problem solving,

the chances of increasing access to multiplicative thinking in the middle years can be greatly enhanced.